

On A Stringy Singular Cohomology

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ABSTRACT

String theory has already motivated, suggested, and sometimes well-nigh proved a number of interesting and sometimes unexpected mathematical results, such as mirror symmetry. A careful examination of the behavior of string propagation on (mildly) singular varieties similarly suggests a new type of (co)homology theory. It has the ‘good behavior’ of the well established intersection (co)homology and L^2 -cohomology, but is markedly different in some aspects. For one, unlike the intersection (co)homology and the L^2 -cohomology (or any other known thus far), this new cohomology is symmetric with respect to the mirror map. Among the available choices, this makes it into a prime candidate for describing the string theory zero modes in geometrical terms.

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1. Introduction and Results

The close relationship between the cohomology of a (Riemannian) space X and the Hilbert space of a supersymmetric σ -model with target space X has been studied and known for well over a decade [1]. In addition, Ref. [1] considered the case of a real, Riemannian manifold X as the target space for an $N = 1$ supersymmetric model. The correspondence derives from the formal isomorphism between the exterior derivative algebra

$$\{d, d^\dagger\} = \Delta, \quad (1.1)$$

and the supersymmetry algebra

$$\{Q, \bar{Q}\} = H, \quad (1.2)$$

and the resulting formal isomorphism between the associated complexes.

With more than one supersymmetry, and on complex manifolds, the above algebras are modified. While the crucial supersymmetry relation (1.2) remains real

$$\{Q_\pm, \bar{Q}_\pm\} = (H \pm p), \quad (1.3)$$

the exterior derivative relation become *holomorphic*

$$\{\partial, \partial^\dagger\} = \Delta_\partial \quad \text{and} \quad \{\bar{\partial}, \bar{\partial}^\dagger\} = \Delta_{\bar{\partial}}. \quad (1.4)$$

Nevertheless, the very high degree of analogy between the two basic relations will ensure the complex generalizations of the results of Ref. [1] in the real case. In fact, on Kähler manifolds, where $\Delta_\partial = \Delta_{\bar{\partial}} = \frac{1}{2}\Delta_d$, we can define $d_\pm \stackrel{\text{def}}{=} \partial \pm \bar{\partial}$, whereupon

$$\{d_\pm, d_\pm^\dagger\} = \Delta_d. \quad (1.5)$$

So, much as the set of zero-modes (a.k.a. the kernel) of Δ_d correspond to the de Rham cohomology and the set of zero-modes of $\Delta_{\bar{\partial}}$ correspond to the Dolbeault ($\bar{\partial}$ -) cohomology, the zero-modes of $(H \pm p)$ correspond to the \bar{Q}_\pm -cohomology¹. Translationally invariant zero-modes (those annihilated by the linear momentum p) are then also annihilated by the Hamiltonian, H , and so have zero energy. Since $\langle H \rangle \geq 0$, these states are the ‘ground states’, i.e., the (supersymmetric) vacua. Technically, it is often easier to consider a ‘twisted’ sibling model in which the \bar{Q}_\pm operators have spin 0 and generate a BRST symmetry. Much as the original supersymmetry, this ‘twisted’ BRST symmetry produces its

¹ To be precise, the translationally invariant zero modes of H are elements of both the \bar{Q}_+ - and the \bar{Q}_- -cohomology. Therefore, ‘the \bar{Q}_\pm -cohomology’ should really be taken as an abbreviation for ‘the \bar{Q}_+ -cohomology \cap \bar{Q}_- -cohomology’. In complex geometry and on a Kähler manifold, this then corresponds to the intersection of the usual $\bar{\partial}$ -cohomology and its conjugate ‘ ∂ -cohomology’.

associated complex and the resulting cohomology, which is in a formal 1–1 correspondence with that of the original supersymmetry. Such ‘twisted’ models are then used to obtain (somewhat restricted) results about the original (‘untwisted’) model, but also to provide a more direct mathematical interpretation because they avoid the complications related to involving spinors.

Most of the literature focuses on cases where X is a smooth manifold. As it turns out however, suitably (and mildly) singular target spaces are indeed of interest [2,3,4,5,6,7] and the relationship between the zero-modes of H , i.e., the \bar{Q}_\pm -cohomology on one hand and the available (co)homology theories on singular varieties on the other ought to be revisited.

The mathematics of singular spaces, and in particular the different types of cohomology on such, is rather well researched but not very well known generally; the Reader is referred to Ref. [8] for an introduction in the general theory. All of the various types of cohomology that can be defined on singular spaces must reproduce the well known results on smooth spaces. However, on singular spaces, these cohomology groups do differ in general.

Partly prompted by this general cohomology theory on singular spaces, and partly by certain assumptions about the supersymmetric σ -model, it was argued [5] that the relevant cohomology on ‘conifolds’² ought to be the so-called intersection (co)homology. The intersection homology is known to be dual to the square-integrable cohomology [8]. Since the cohomology representatives correspond to supersymmetric vacuum wave-functions [1], the square-integrability appears to be a natural condition. In addition, conifolds appear as interfaces in processes of topology change of Refs. [2,3], and the intersection cohomology appeared to be a good choice (see below).

The purpose of this note is to describe and provide a *working definition* for a new type of cohomology for singular spaces. This new cohomology arises from the recent results of Refs. [6,7] which presents a novel characteristic of superstring models with singular target spaces. Not surprisingly, this new cohomology satisfies the so-called ‘Kähler package’ requirements. This is exactly as with the intersection (co)homology and the square-integrable cohomology — which had been one of the main reasons in Ref. [5] for their use in discussing strings on singular spaces.

However, unlike any other (co)homology theory known to the present author, this new cohomology is perfectly symmetric with respect to the still conjectured but increasingly better accepted ‘mirror map’ [11,12]. This we take as the essential bit of evidence that this new cohomology is *the* cohomology of string theory.

² The term ‘conifold’ was invented in the last of the works in Ref. [3], where it was defined rather loosely. The models examined there only contained nodes. However, a rather larger but still not very well specified class of singularities ought to be included [4], and we will understand ‘conifolds’ to do so. Furthermore, the recent developments [7] indicate that the category of spaces ought to be generalized to include ‘stratified pseudomanifolds’ [9,10] (for a definition, see Ref. [8]).

2. The Stringy Singular Cohomology

Mostly for the sake of simplicity, we will consider the case where the singular model, X^\sharp is smooth apart from a finite number of isolated nodes. Given Arnold's classification of singularities [13] and their small resolutions³ [14], it should not be too difficult to generalize the present discussion to all isolated (A, D, E) singularities.

2.1. The conifold transition

We recall from [2,5] (see Ref. [14] for a more general discussion) the basic process which connects topologically distinct Calabi-Yau target spaces. We start with a smooth Calabi-Yau manifold X_ϵ^\flat the complex structure of which is made to depend on a complex parameter ϵ in such a way that for sufficiently small but nonzero ϵ the space X_ϵ^\flat is smooth, but $\lim_{\epsilon \rightarrow 0} X_\epsilon^\flat = X^\sharp$ is singular. Let X_ϵ^\flat acquire n nodes as $\epsilon \rightarrow 0$. The singular model X^\sharp then admits a *small resolution*, \tilde{X} , which again is a smooth Calabi-Yau manifold and has a \mathbb{P}^1 in place of every node of X^\sharp . The transition $X_\epsilon^\flat \rightarrow X^\sharp \rightarrow \tilde{X}$ is but the simplest type of a conifold transition; in the others types, X^\sharp has singularities worse than nodes, and accordingly, \tilde{X} will be a more complicated small resolution⁴.

For $\epsilon \neq 0$, the homology group $H_3(X_\epsilon^\flat)$ admits a symplectic decomposition (perhaps depending on ϵ) into A_i and B^i 3-cycles, with $i = 0, \dots, (\frac{1}{2}b_3 - 1)$, such that:

$$A_i \cap A_j = 0, \quad B^i \cap B^j = 0, \quad \text{and} \quad A_i \cap B^j = \delta_i^j. \quad (2.1)$$

Locally, as $\epsilon \rightarrow 0$, each node is seen to arise as an S^3 -shaped 3-cycle (one of the A_i 's) that shrinks to the singular node when $\epsilon = 0$. While $\epsilon \neq 0$, each of these S^3 's is an element of $H_3(X_\epsilon^\flat)$. However, not each of the n S^3 's need represent an independent homology element A_i . Suppose that there is a single 4-chain (an equivalence class of real 4-dimensional objects), C_ϵ , of which the n S^3 's comprise the boundary. Then a formal sum of the n S^3 's (with the signs dictated by the relative orientation) is homologically trivial, and the last of the n S^3 's can be expressed as the negative of the sum of the other $(n-1)$ S^3 's. This leaves $(n-1)$ independent A_i 's, which we will label \hat{A}_i , the $(n-1)$ independent elements of $H_3(X_\epsilon^\flat)$ represented by the n S^3 's. Finally, since these n S^3 's form $(n-1)$ independent and non-intersecting cohomology elements (a subset of the A_i 's), there must exist $(n-1)$ duals to these: $(n-1)$ of the B^i 's, which we will label \hat{B}^i .

³ Unlike blow-ups, small resolutions *always* leave the Calabi-Yau condition $c_1 = 0$ intact.

⁴ Strictly speaking, $\tilde{X} \rightarrow X^\sharp$ must be a crepant desingularization: a holomorphic isomorphism except at the inverse images of the singular points of X^\sharp , and which contributes nothing to the canonical divisor. Thus, the whole process involves only Calabi-Yau models.

When $\epsilon \rightarrow 0$, the n ‘boundary’ S^3 ’s shrink to points, whereby $C_\epsilon \rightarrow C_0$ becomes a proper cycle. Thus, in X^\sharp , there is one new 4-cycle C_0 , and in addition, there remain the $(n-1)$ 3-cycles \hat{B}^i . Now, in the intersection homology theory, the $(n-1)$ \hat{B}^i ’s are not counted as proper non-trivial homology elements. In the square-integrable cohomology theory, one can show that the corresponding forms are no longer square-integrable.

While somewhat deceiving because of low dimension, the example of a pinched torus is perhaps instructive. When a torus is deformed so that one of the ‘small circles’ shrinks to a point, all small circles become homologous to this point. This point is a singularity of the pinched torus. Formally, the (middle-dimensional) intersection homology is determined by excising the singular points, which now turns the $(n-1)$ 3-cycles \hat{B}^i into chains with boundaries, whereupon these become trivial in homology. This is how intersection homology loses both the $(n-1)$ 3-cycles \hat{A}_i represented by the n S^3 ’s that have shrunk to the singular points, and also their $(n-1)$ dual 3-cycles \hat{B}^i . In the square-integrable cohomology, the forms supported on \hat{B}^i become divergent in the limit when the node forms. In the pinched torus case, this is clear on noting that the pinching of a cylindrical section is equivalent to stretching it thinner and longer.

Among 4-cycles, there is the newly formed C_0 . In the intersection homology it acquires a dual 2-cycle—traced to the limiting boundaries of the $(n-1)$ chains created from the 3-cycles C^3 upon the excision of the n singular points. These limiting objects however do not exist in the usual homology where the singular points are not excised. This is how intersection homology of X^\sharp gains the one dual pair of 4- and 2-cycles while losing $2(n-1)$ 3-cycles, which makes it formally isomorphic to the homology of the small resolution \check{X} . In a sense, this is the reason for the asymmetry of the intersection homology as will be noted more clearly below.

Finally \check{X} , a small resolution of X^\sharp , consists of replacing the n nodes with S^2 ’s, all of which meet the C_0 in a single point and are therefore all dual to that one 4-cycle C_0 . Thus, the n S^2 ’s are all homologous and represent a single 2-cycle. This is how small resolution provides a 2-cycle in the usual homology in place of the formal 2-cycle in the intersection homology of the singular space. At the same time, each of the n S^2 ’s opens a single ‘hole’ in the $(n-1)$ \hat{B}^i ’s, which thereby become chains and are no longer non-trivial homology elements. Again, this is how small resolution loses the 3-cycles remaining after the n S^3 ’s have shrunk to points, and which intersection homology formally discards.

This leaves none of the $(n-1)$ \hat{A}_i ’s and \hat{B}^i ’s from X^\flat surviving into \check{X} . On the other hand, the shrinking of the n S^3 ’s has made C_0 into a proper 4-cycle and the process of small resolution of the nodes has created one dual 2-cycle (homology class) of the S^2 ’s.

Clearly, had we started with m relations among the n S^3 , there would have been m \hat{C}_0^a ’s and the n S^2 ’s would have formed m homology classes \hat{D}_a in \check{X} . This more general situation is then presented in Table 1 (see also Ref. [5]).

k	$\dim' H_k(X^\flat)$	$\dim' H_k(X^\sharp)$	$\dim' IH_k(X^\sharp)$	$\dim' H_k(\check{X})$
2	0	0	m	m
3	$2(n-m)$	$(n-m)$	0	0
4	0	m	m	m

Table 1: A chart of the varying contributions to the ‘standard’ homology and intersection homology (IH_*) for X^\flat , X^\sharp and \check{X} .

It should be obvious that $H_k(X^\sharp)$ (or its dual cohomology) cannot possibly admit a Hodge decomposition, since the number of 3-cycles may well be odd. Neither can $H_k(X^\sharp)$ (or its dual cohomology) admit Poincaré duality since the $m \hat{C}_0^a$ ’s in H_4 have no duals in H_2 . Both of Hodge decomposition and Poincaré duality are crucially related to well known properties of the corresponding σ -models⁵ and so must exist in any candidate cohomology. The intersection homology certainly does admit both Hodge decomposition and Poincaré duality [8], and so is a viable candidate for a cohomology describing the supersymmetric vacua in a corresponding supersymmetric σ -model. (These then correspond to massless fields in the effective spacetime field theory.)

Note that $IH_k(X^\sharp)$ is identical to $H_k(\check{X})$ while being very different from $H_k(X^\flat)$ —this is the asymmetry mentioned above.

2.2. Massless black holes and homology

The conifold transition $X^\flat \rightarrow X^\sharp \rightarrow \check{X}$ is still not fully understood as a physical phase transition between target spaces for the heterotic string. However, Type II superstrings have twice as much spacetime supersymmetry, which enabled the uncovering of a physical mechanism that achieves a trouble-free interpolation between the target spaces X^\flat , X^\sharp and \check{X} [6,7]. The results are merely retold in the sequel; the Reader is referred to Ref. [6,7] for further details of the physics mechanism in this phase transition.

The $N = 2$ spacetime supersymmetry of Type II strings permits the existence of (BPS) states which preserve half of the supersymmetry, and the mass of which is bounded (and in

⁵ Hodge decomposition is related to the existence of both left- and right-moving fermions and the associated left-right symmetry of the 2-dimensional $(2,2)$ -supersymmetric field theory of strings. In $(0,2)$ -models which have no left-right symmetry, the relevant cohomology is valued in a vector bundle (or sheaf) other than the (co)tangent bundle. Poincaré duality corresponds to CPT conjugation which is a symmetry in the underlying 2-dimensional relativistic field theory.

the extremal case equal) to their charge. Such states accompany the not-yet-shrunk $(n-m)$ \hat{A}_i 's of X_ϵ^b and their mass is proportional to the volume of the S^3 's representing these cycles. The effective field theory of the Type IIB strings contains a 4-form potential. Field configurations where the field strength (5-form) F has a non-zero integral over the 3-cycle \hat{A}_i within the Calabi-Yau 3-fold appear, in the complementary 4-dimensional Minkowski space, as monopoles with the field strength (2-form) $\int_{\hat{A}_i} F$.

As the S^3 's shrink to points in the $\epsilon \rightarrow 0$ limit, these states become massless. So, the effective field theory derived from a Type II string theory constructed on X_ϵ^b loses $(n-m)$ ‘elementary’ massless states as $\epsilon \rightarrow 0$, but promptly gains their replacements in the form of these by now massless BPS states. Thus, the low energy effective physics counts $2(n-m)$ massless states throughout the variation of ϵ , from nonzero values to *and including* $\epsilon = 0$.

Furthermore, there also exists m states corresponding to the m relations among the n nodes, and these states essentially provide a ‘mirror story’. They are massive and have nonzero vacuum expectation values in the model built upon \check{X} , but become massless as the n S^2 's are shrunk to points. Recall that these n S^2 's form m 2-cycles, \hat{D}_a , which are dual to the 4-cycles \hat{C}_0^a , as described above. The m BPS states then can be seen to stem from field configurations of the 3-form potential in the Type IIA theory, the field strength G of which have non-zero integrals $\int_{\hat{D}_a} G$, and which then appear as field strength 2-forms of monopoles (one for each \hat{D}_a) in the 4-dimensional Minkowski spacetime. Again, each of these m states has a mass proportional to the area of the S^2 's representing its 2-cycle.

Thus, in the conifold transition, the varying set of massless fields—with these massless BPS states included!—is in a sense complementing the situation presented in Table 1, and the results are summarized in Table 2.

k	$\dim' H^k(X^b)$	$\dim' H^k(X^\sharp)$	$\dim' SH^k(X^\sharp)$	$\dim' H^k(\check{X})$
2	0	0	m	m
3	$2(n-m)$	$(n-m)$	$2(n-m)$	0
4	0	m	m	m

Table 2: A table of the varying contributions to the massless field spectrum in the effective field theory of Type II superstrings built upon X^b , X^\sharp and \check{X} . The states are listed as corresponding to cohomology groups.

Notice that rather than being equal either $H^*(X^b)$ or $H^*(\check{X})$, this set of new cohomology groups $SH^*(X^\sharp)$ are in fact, formally, the *union* of $H^*(X^b)$ and $H^*(\check{X})$! It is precisely these $SH^*(X^\sharp)$ that are herein proposed as the (super)stringy singular cohomology⁶.

⁶ The acronym SSC, albeit naturally formed, is perhaps best avoided for being ominous in view of the tightness of the present funding situation.

2.3. Calculating the (super)stringy singular (co)homology

No general cohomology theory for singular spaces that would produce this new cohomology group is known as yet to the present author. However, modeled on the Corollary on p. 313 of Ref. [15], a *working definition* of the (super)stringy singular homology, $SH_*(X^\sharp)$ is provided below. The corresponding cohomology, $SH^*(X^\sharp)$, is then defined as the formal dual. Some rough characteristics of the elements of $SH^*(X^\sharp)$ by which it differs from (the square-integrable) $H_{(2)}^*(X^\sharp)$ will also transpire.

For practical purposes of calculating the intersection homology of the singular limit $X^\sharp = \lim_{\epsilon \rightarrow 0} X_\epsilon^\flat$, we quote the Corollary from Ref. [15] (see also § 7.3 of Ref. [5]):

Corollary

Let X be an n -fold with a single isolated singularity, x . Then

$$IH_k(X) = \begin{cases} H_k(X) & k > n, \\ \text{Im}[H_n(X-x) \rightarrow H_n(X)] & k = n, \\ H_k(X-x) & k < n. \end{cases}$$

For practical purposes of calculating the (super)string singular homology of the singular limit $X^\sharp = \lim_{\epsilon \rightarrow 0} X_\epsilon^\flat$, we then propose

Definition

Let X be an n -fold with a single isolated singularity, x . Then

$$SH_k(X) = \begin{cases} H_k(X) & k > n, \\ H_n(X-x) \cup H_n(X) & k = n, \\ H_k(X-x) & k < n. \end{cases} \quad (2.2)$$

The definition generalizes to cases where the singular locus, x , consists of more than one point and in fact of several component subspaces of X —just as it does in the case of the above corollary for IH_k . Also, the formal union in (2.2) may be better understood as an extension of $H_n(X)$ by $\ker[H_n(X-x) \rightarrow H_n(X)]$, or perhaps of $H_n(X-x)$ by $\text{coker}[H_n(X-x) \rightarrow H_n(X)]$. The precise nature of this extension is to be determined from the as yet unspecified general cohomology theory.

We note that both the above corollary and the above *working definition* provide a prescription for calculating IH_k and SH_k , and in terms intrinsic to the singular space itself and independent of any possible smoothings (through deformation, or any resolution). We therefore *expect* that a well-defined and fully intrinsic general homology theory for SH_k can also be found, just as there is one for IH_k [8]. This task is however beyond our present scope.

The situation with the ring structure of SH_* is somewhat different. Since dual pairs \hat{A}_i, \hat{B}^j in the middle dimensional homology $SH_n(X^\sharp)$ come from different spaces in the definition, $H_n(X-x)$ and $H_n(X)$ respectively, their intersection ring structure is not as well defined as for the rest of the n -cycles which pass through the conifold transition unaffected. It seems however reasonable to consider the ring structure of $H_n(X_\epsilon^\flat)$ as a function of ϵ , and assign the limit $\epsilon \rightarrow 0$ of this ring structure to $SH_n(X^\sharp)$. Similarly, the part of $SH_k(X^\sharp)$ with $k > n$ or $k < n$, represented by the \hat{C}_0^a, \hat{D}_a and which is absent from $H_k(X_\epsilon^\flat)$, is equally difficult to deal with. These cohomology elements do have their counterparts in $H_k(\check{X})$ and a well-defined ring structure there—both the classical (generated from the wedge product) and also the quantum one (deformed as a function of the family of rational curves in \check{X}). Let ψ denote the root-mean-square area of the S^2 of \check{X} which replaced the nodes of X^\sharp . Then, the limit $\psi \rightarrow 0$ of the ring structure of $H_k(\check{X}_\psi)$, for $k \neq 0$, may be assigned as the ring structure of $SH_k(X^\sharp)$.

In other words, each Calabi-Yau space is fibred over its respective space of complex structures and extended and complexified Kähler class (the latter defined more precisely in Ref. [9]). The smooth spaces X_ϵ^\flat and \check{X}_ψ each occur in such a family (although only the parameters relevant for the conifold transitions are noted explicitly). In either of these families the (quantum) ring structures of $H^{3-*,*}$ and of $H^{*,*}$ depend on the moduli parameters. The singular space(s) X^\sharp appear on the joining interface between the X_ϵ^\flat family and the \check{X}_ψ family, and so acquire a cohomology ring structure in the limit.

These assignments obviously depend on the choice(s) of the smoothing deformation $X^\sharp \rightarrow X_\epsilon^\flat$ and the choice(s) of resolution of singularities $X^\sharp \rightarrow \check{X}_\psi$. This dependence on deformation i.e. resolution is perhaps undesirable as it is not intrinsic to the singular space X^\sharp itself. However it seems to make perfect sense when considering the connected families of spaces $X_\epsilon^\flat, X^\sharp, \check{X}_\psi$ fibered over the joined total moduli space (both the space of complex structures and the extended and complexified Kähler classes [2,3,5,16]).

3. Some Notable Properties

Besides being simply another set of (co)homology groups defined on (certain) singular spaces, SH_* and SH^* exhibit a few properties by which they are distinguished from the rest. We now turn to examine two: relation to mirror symmetry and the properties commonly grouped under the name ‘Kähler package’.

3.1. Mirror symmetry

Consider a conifold transition $M_\epsilon^b \rightarrow M^\sharp \rightarrow \check{M}_\psi$ and recall that in this process b_3 decreases while b_2+b_4 increases. Consider now the corresponding families of mirror spaces $W_\epsilon^b, W^\sharp, \check{W}_\psi$. The mirror map exchanges (the rôles in the cohomology valued in) the tangent and the cotangent bundle, whereby b_3 and b_2+b_4 are exchanged. Therefore, in the mirror of a conifold transition, the Betti numbers change in the opposite direction. Assuming that the spaces $W_\epsilon^b, W^\sharp, \check{W}_\psi$ are also connected through a conifold transition, we then have a mirror pair of conifold transitions:

$$M_\epsilon^b \rightarrow M^\sharp \rightarrow \check{M}_\psi , \quad (3.1a)$$

$$\check{W}_\psi \rightarrow W^\sharp \rightarrow W_\epsilon^b . \quad (3.1b)$$

Consider now using IH_k for all of these spaces (in the smooth cases $IH_k = H_k$, by definition). By virtue of the results in Table 1, we obtain the rather asymmetric pair:

$$IH_k(M_\epsilon^b) \not\cong IH_k(M)^\sharp \cong IH_k(\check{M}_\psi) , \quad (3.2a)$$

$$IH_k(\check{W}_\psi) \cong IH_k(W)^\sharp \not\cong IH_k(W_\epsilon^b) . \quad (3.2b)$$

By contrast, using the results in Table 2, SH_k produces a perfectly mirror-symmetric pair of conifold transitions:

$$SH_k(M_\epsilon^b) \subsetneq SH_k(M)^\sharp \supsetneq SH_k(\check{M}_\psi) , \quad (3.3a)$$

$$SH_k(\check{W}_\psi) \subsetneq SH_k(W)^\sharp \supsetneq SH_k(W_\epsilon^b) . \quad (3.3b)$$

For the smooth spaces $SH_k = H_k$, by definition.

Also, $SH_k(X^\sharp)$ is bigger than either $H_k(X^b)$ or $H_k(\check{X})$ — in good agreement with the physics intuition that string theory on X^\sharp ought to possess extra zero-modes, stemming from extra symmetries (continuous or discrete) and forced by anomaly cancellation of these symmetries. This general expectation is indeed borne out in the analysis of Type II strings [6,7], the analysis of which was facilitated by the $N = 2$ supersymmetry. In the heterotic string case, the $N = 1$ supersymmetry does not provide enough rigidity to derive analogous results, and in addition, the heterotic string lacks the 4- and 3-forms of the Type II strings the topologically non-trivial configurations of which provide for the BPS (monopole-like) states. As stated in Ref. [6], the physical mechanism of the conifold transition remains a mystery for the heterotic strings.

3.2. The Kähler package

One of the major arguments in favor of using IH_k (and its formal dual IH^k) for singular spaces in Ref. [5] was the fact that this (co)homology features the properties collectively referred to as ‘the Kähler package’:

1. Hodge decomposition:

$$H^r(X) = \bigoplus_{p+q=r} H^{p,q}(X) ; \quad (3.4)$$

2. Complex conjugation:

$$\overline{H^{p,q}(X)} = H^{q,p}(X) ; \quad (3.5)$$

3. Poincaré duality:

$$H^{p,q}(X) \overset{*}{\approx} H^{n-q,n-p}(X) ; \quad (3.6)$$

4. Künneth formula:

$$H^{r,s}(X \times Y) = \bigoplus_{\substack{p+p'=r \\ q+q'=s}} H^{p,q}(X) \otimes H^{p',q'}(Y) . \quad (3.7)$$

5. Lefschetz $SL(2, \mathbb{C})$ action (with ω the Kähler form on the n -fold X):

$$\begin{aligned} L(\eta) &\stackrel{\text{def}}{=} \omega \wedge \eta , & \Lambda &\stackrel{\text{def}}{=} (-)^n * L * , & \forall \eta \in H^{*,*}(X) , \\ h &\stackrel{\text{def}}{=} [L, \Lambda] , & [h, L] &= L , & [h, \Lambda] &= -\Lambda , \end{aligned} \quad (3.8)$$

and the induced decomposition into irreducible $SL(2, \mathbb{C})$ representations (the so-called ‘Lefschetz Hard Theorem’).

The first four parts of the Kähler package have immediate counterparts in the 2-dimensional field theory underlying the string propagation on X . These are easily seen to be featured by both IH_k and by SH_k .

On Calabi-Yau spaces, the last property may literally be doubled [17] owing to mirror symmetry. That is, *there really exist two ‘orthogonal’ Lefschetz $SL(2, \mathbb{C})$ actions*. The conventional one acts ‘vertically’, whereas the second one acts ‘horizontally’ in the Hodge diamond. While the conventional one is generated by wedge products with the Kähler form (3.8), the second $SL(2, \mathbb{C})$ action is the mirror-preimage of the conventional Lefschetz $SL(2, \mathbb{C})$ action on the cohomology of the mirror space. Both $SL(2, \mathbb{C})$ actions induce a decomposition into irreducible $SL(2, \mathbb{C})$ representations. Finally, at least for Calabi-Yau 3-folds, these two $SL(2, \mathbb{C})$ actions commute, and rather trivially, since the cohomology elements not annihilated by one action are annihilated by the other. While the physics

application and implication of these actions is not yet clear, requiring their existence poses rather severe restrictions on the cohomology groups, and cohomology theory in general.

Just as with the discussion of the ring structure developed on SH_k , we again define the ‘vertical’ and ‘horizontal’ Lefschetz $SL(2, \mathbb{C})$ action on X^\sharp as a limit from \check{X}_ψ and from X_ϵ^\flat , respectively. While not satisfactory as an intrinsic definition, this certainly makes perfect sense for the connected families of $X_\epsilon^\flat, X^\sharp, \check{X}_\psi$.

In fact, already with $IH^k(X^\sharp)$, one can trace the conventional (vertical) $SL(2, \mathbb{C})$ action as the limit of the same on $H^k(\check{X}_\psi)$, to which it is isomorphic (for any ψ) as a vector space. Comparing tables 1 and 2, we see that, on an n -fold, $IH_* \subseteq SH_*$ and $IH_k \subset SH_k$ only for $k = n$; the analogous is true for the formal duals IH^k and SH^k . So, except for the middle dimension, the proof of the conventional Lefschetz $SL(2, \mathbb{C})$ action for IH^k [8] will also apply for SH^k . On Calabi-Yau 3-folds—the case of our immediate interest—all middle dimensional cohomology is annihilated by the raising and lowering operators of $SL(2, \mathbb{C})$ simply because the $(3-q\pm 1, q\pm 1)$ cohomology is empty. This then obviously holds both for $IH^3(X^\sharp)$ and $SH^3(X^\sharp)$. The argument for the ‘horizontal’ $SL(2, \mathbb{C})$ action is precisely analogous, upon exchanging the rôles of the rows and columns of the Hodge diamond, and the limit obtained from X_ϵ^\flat by letting $\epsilon \rightarrow 0$.

Both of the $SL(2, \mathbb{C})$ actions are expected to degenerate partly in the singular limit $\epsilon, \psi \rightarrow 0$, since non-degeneracy is guaranteed only for generic models [17]. At the location of such degeneration, some of the the irreducible $SL(2, \mathbb{C})$ representations decompose into smaller ones. For example, the isomorphism $\omega_\psi : H^{1,1}(\check{X}_\psi) \xrightarrow{\sim} H^{2,2}(\check{X}_\psi)$ for a family of 3-folds⁷ is realized by wedging with the ψ -dependent Kähler form ω_ψ . It pairs a $(1,1)$ -form with a $(2,2)$ -form into an $SL(2, \mathbb{C})$ doublet. At special values ψ' , this map will develop a non-zero kernel and cokernel. Some of the $(1,1)$ -forms that are now annihilated by $\omega_{\psi'}$, and a corresponding number of $(2,2)$ -forms are no longer obtainable as $\omega_{\psi'} \wedge \eta$ for $\eta \in H^{1,1}(\check{X}_{\psi'})$. These unpaired $(1,1)$ - and $(2,2)$ -forms have now become pairs of 1-dimensional (trivial) representations of $SL(2, \mathbb{C})$. Ref. [17] gives concrete examples for the degeneration the ‘horizontal’ Lefschetz $SL(2, \mathbb{C})$ action, but the same applies quite clearly for the standard (‘vertical’) action also.

In a way, the (co)homology SH_* (SH^*) may be considered the answer to the question if there is a (co)homology on (at least some) singular spaces, and other than IH_* (IH^* and the L^2 -cohomology), that features the Kähler package.

⁷ Note: ψ parametrizes the choice of the Kähler form, not the complex structure. However, we fiber the Calabi-Yau spaces over the combined space of complex structures and complexified and extended Kähler classes as the moduli space.

3.3. Some open considerations

Besides the Hodge decomposition of the cohomology (the first point in the Kähler package), the \bar{Q}_\pm -cohomology of the 2-dimensional σ -model also exhibits the Hodge decomposition of forms. This amounts to the statement that any cohomology element is represented by a harmonic form (annihilated by $H \pm p$) up to the addition of a \bar{Q}_\pm -exact and a Q_\pm -exact term. It would be interesting to see if this property extends, perhaps with respect to the d_\pm operators defined for Eq. (1.5), to SH^* .

The definition (2.2) may well appear to be rather *ad hoc* and it is tempting to speculate if it could be derived from the underlying structure of string theories. The two natural approaches would involve either the (canonical, Hamiltonian) configuration space of string theory, the loop space $\mathcal{L}X$, or the (relativistically covariant, Lagrangian) configuration space, the space of maps $\Sigma \rightarrow X$, where Σ is the universal curve.

At least in the $n = 1$ case (tori), the loop-space approach suggests a physical reason for retaining the 1-cycles represented by the ‘small’ circles even after one of them has been shrunk to a point. Possessing a non-zero tension, the string simply cannot be made to shrink to the singular point. This, of course is also consistent with the $R \leftrightarrow \frac{1}{R}$ duality — one of the first duality relations found in string theory. The situation in $n > 1$ is then likely to be more involved and will perhaps relate to some of the more recently uncovered duality relations.

Just as the relation to mirror symmetry seems to be a strong indication in favor of SH_k , the latter approach introduces another possibility to relate SH_k to at least some of the newly discovered dualities of string theory. To that end, note that maps $\Sigma_g \rightarrow X$, with Σ_g a Riemann surface of genus g , may be thought of as graphs in the space $\Sigma_g \times X$, whence string theory may be understood in terms of a field theory in the 12-dimensional $\Sigma_g \times X$ constrained however to the graphs of $\Sigma_g \rightarrow X$. In turn, it is becoming clearer that the ‘master theory’ for the dualities is probably a 12-dimensional theory the lower dimensional limits of which exhibit local supergravity. Yet, this 12-dimensional theory (‘F-theory’) itself cannot be a supergravity theory since the (smallest) spinor representation is too large to be mapped, 1–1, to tensorial representations of (integral) spin 2 and less. Therefore, this 12-dimensional theory must be a constrained $N = \frac{1}{2}$ theory, and may well be describing the graph of the string in the 12-dimensional space $\Sigma_g \times X$. In this constrained 12-dimensional theory and in particular on its cohomology, the duality involutions should have a simpler action (much as the non-linear Möbius transformation becomes linear when acting on the homogeneous coordinates of \mathbb{P}^1). This could provide additional clues about the cohomology of this master theory, and so also about the induced cohomology of the target space of the string propagation — hopefully, the SH^* defined herein or its generalization.

In closing, a possible mathematical framework in which to define intrinsically the SH^* and SH_* groups appears natural to suggest. Recall that the ‘stringy cohomology’

on Coxeter orbifolds (T^6/D , where D is the action of a finite group) introduced ‘twisted states’: string configurations which are contractible in conventional geometry, but are obstructed from contracting by the string dynamics (tension). Such cohomology elements are localized at the finite quotient singularities of these orbifolds [18]. In this case, the mathematical framework for a rigorous definition of this cohomology turned out to be equivariant K -theory [19], owing to the fact that such orbifolds are global quotients by a finite group G . The present case is more complicated, because the singularities now do not stem from a *global* quotient by the non-free action of a discrete group G . Consequently, there are no G -bundles to define a suitable K -theory and a G -equivariant cohomology. Instead, there exist local group actions, restricted to the singular points and the tangent cones centered at the singular points. Instead of a global G -bundle for a K -theory, then, sheaves with only locally supported group actions seem natural candidates, leading to a suitably generalized, SK -theory—hopefully the underlying theory for a rigorous definition of SH^* and SH_* .

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